Whispering Gallery Orbits in the Bunimovich Stadium

C. A. Kruelle[a], A. Kittel[b], J. Peinke[b], and R. Richter[c]

[a] Max-Planck-Institut für Metallforschung, Heisenbergstraße 1, D-70569 Stuttgart
[b] Physikalisches Institut, Universität Bayreuth, D-95440 Bayreuth
[c] Universität Magdeburg, Institut für Experimentelle Physik,
 Abteilung Nichtlineare Phänomene, D-39016 Magdeburg

A new visualization method is found for investigating the complex ray dynamics of whispering gallery orbits inside the Bunimovich stadium. Regarding the whispering gallery as mirror cabinet, a fractal system of virtual images of the mirror walls is found which reflects the sensitive dependence of the dynamic behavior on the initial conditions.

Key words: Chaotic systems; fractals; optics; whispering gallery; Ray tracing.

1. Introduction

In 1904 Lord Rayleigh noticed in the dome of St. Paul's Cathedral in London that sound waves are propagated particularly easily along the inner surface of a concave wall [1, 2]. If a person who is situated at position B in Fig. 1 whispers along the wall, another person at position F can hear his voice clearly.
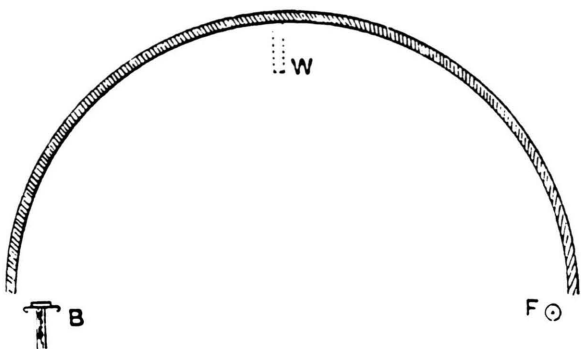

Fig. 1. Lord Rayleigh's experimental setup for the investigation of the whispering gallery. B: bird-call, F: flickering flame, W: wooden bar, (from [2]).

Rayleigh noticed that this is not due to a focusing effect as in, for example, a parabolic mirror. Here, the sound waves are "guided" by multiple reflections

along the semicircle of the wall. The effect is enhanced if the speaker whispers, since the wavelength of a high-pitched voice is shorter and therefore the main beam of the radiation pattern "irradiated" from the finite aperture of the mouth becomes narrower. Rayleigh investigated this phenomenon in his laboratory, using a bird-call B ($\lambda = 2$ cm) as source and a flickering flame F as detector inside a twelve feet long strip of zinc. His interpretation was supported by the fact that a direct barrier between the bird-call and the flame did not destroy the flicker, while a small wooden bar at position W did.

Since the guidance of waves along the inner surface of a circular boundary was first discovered inside such a whispering gallery, the corresponding ray trajectories were named "whispering gallery orbits".

All trajectories inside the circle propagate, depending on the initial conditions, along periodic or quasiperiodic orbits, respectively. The initial angular momentum of any ray is conserved. In contrast, for example, the Bunimovich stadium (two semicircles separated by a square) is a system whose trajectories can be classified into a discrete set of periodic orbits of measure zero separated by an innumerable number of chaotic orbits [3, 4]. Fig. 2a shows a typical chaotic trajectory.

However, in the stadium exists a system of whispering gallery orbits as well. The first three specimens of an infinite, discrete set of *periodic* whispering gallery orbits inside the stadium can be seen in Fig. 2b. The underlying construction rule is easily found: The
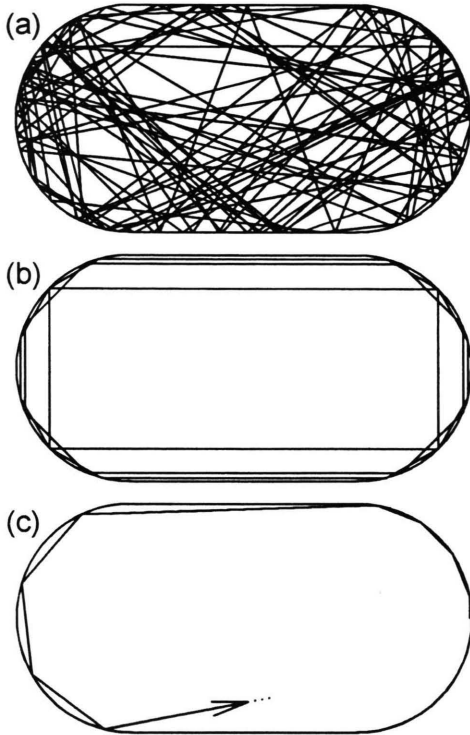
Fig. 2. (a) Chaotic trajectory in the Bunimovich stadium, (b) 3 periodic whispering gallery orbits, (c) nonperiodic whispering gallery orbit.
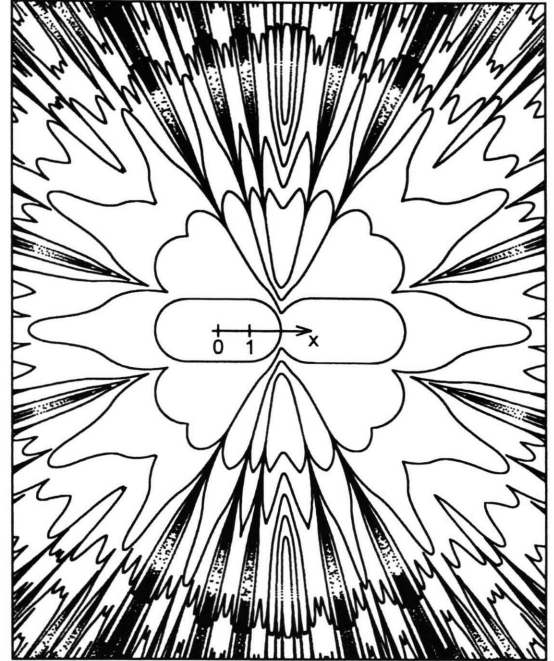

Fig. 3. Section of the system of virtual image walls as seen by an observer inside a mirror cabinet with a Bunimovich stadium shape who is situated at position (1.98, 0). The pronounced channels correspond to the whispering gallery orbits inside the stadium.

regular $4n$-polygons (square, octagon, and so on) are cut in half and connected by two straight lines parallel to the flanks of the stadium.

In addition to these periodic orbits, an innumerable number of trajectories exists that are not periodic but guided along the curves of the stadium as well (see Figure 2c). For the whisperer remains the problem: How can he find the initial direction for sending a message to the listener on the other side of the stadium?

2. Mirror Cabinet Map

To solve this problem the recently introduced "mirror cabinet map" [5, 6] can be applied. We consider a whisperer who is situated at an emission point $(x, 0)$ very close to the boundary and sends out his message into a chosen direction φ. The stadium is seen as a mirror cabinet. According to reflection laws, the initial sound ray will be multiply reflected at the mirror cabinet walls. Using a ray tracing technique we determine the path lengths between two consecutive

reflections. The obtained sequence of path lengths is plotted along a straight line departing from the whisperer into the initial direction φ ignoring the changes of direction by the reflections at the boundary. Repeating this procedure for all possible initial directions φ one obtains a system of virtual images of the stadium wall, *outside* of the boundary.

Figure 3 shows the result. The sensitive dependence of the ray trajectories on the initial conditions is reflected by the increasing roughness of the image walls. The geometric structure of the virtual images develops into a multifractal [6]. Here, the whisperer is situated at the position (1.98, 0) very close to the right end of the stadium. Outside the mirror wall he can observe 7 pointed "channels" above and below the x-axis, respectively. They correspond to trajectories inside the stadium that are injected into the left curve with a grazing angle and will slide in the following along the inner surface of the curve before being reflected into the stadium again (see Figure 2c). A message sent by the whisperer into one of these 14 directions will reach the listener on the other end of the stadium via a whispering gallery orbit.
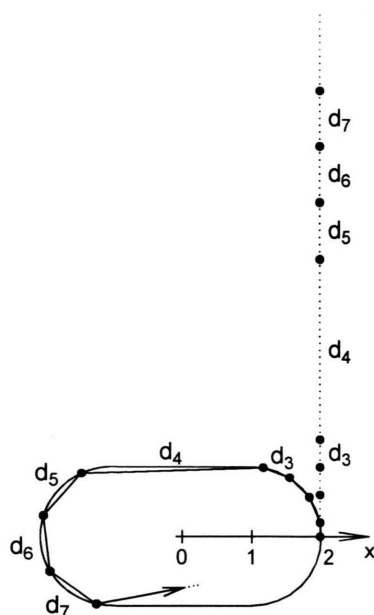
Fig. 4. Principle of the whispering gallery map, detailed description in the text.

The number of such suitable directions depends heavily on the distance of the whisperer to the wall. To be more specific we have to examine the series of *periodic* whispering gallery orbits (Fig. 2b) more carefully. Since they are constructed by inscribing regular $4n$-polygons into the semicircles of the stadium, it can easily be shown that the intersection of the nth periodic orbit with the x-axis is at the position $x_n = 1 + \cos(\pi/4n)$. Applying the mirror cabinet map for different starting positions $(x, 0)$ on the x-axis, one can observe the following:

• For $x \lesssim x_2 = 1.9238\ldots$ the number of "whispering gallery channels" is 6: three above and three below the x-axis, respectively.

• In the regime $x \in [x_2 - 0.005, x_2 + 0.005]$ the "nucleation" and separation of four more channels takes place.

• For $x_2 \lesssim x \lesssim x_3 = 1.9659\ldots$ we have 10 channels.

• For $x_3 \lesssim x \lesssim x_4 = 1.9807\ldots$ there are 14 channels, and so on.

Since the number and the initial directions of the whispering gallery orbits found by this method depends so much on the distance $(2 - x)$ of the whisperer

from the wall, it would be desirable to have a tool for mapping this dependence of the trajectory dynamics on the starting *position* as well.

3. Whispering Gallery Map

If the mirror cabinet idea is slightly varied, all whispering gallery channels reachable from the x-axis can be plotted into one single diagram. We coin this variation the "whispering gallery map". Figure 4 demonstrates the principle. Here, the initial direction φ of the departing ray is fixed. The whisperer is always facing upwards. On the other hand, his position inside the stadium is variable, that is, he can slide along the x-axis. For starting positions x near the right end of the stadium it becomes possible that the emitted sound ray is projected into the opposite half of the stadium with a grazing angle and is therefore forced to propagate along the curve. Using a ray tracing technique, the path lengths d_i between two consecutive reflections are determined. The obtained sequence of path lengths is plotted along the initial direction $\varphi = \pi/2$. Repeating this procedure for all possible starting positions x one gets a system of curves $s_n(x) = \sum_{i=0}^{n} d_i(x)$, where $s_n(x)$ is the total path length, up to the nth reflection, of a ray that departs from the starting point $(x, 0)$.

Figure 5 shows the result of this mapping technique for starting positions $x \in [1.98, 2]$ very close to the boundary. We obtain an infinite series of "tulip"-like channels that converges to the boundary at $x = 2$. The corresponding whispering gallery orbits are not periodic. The trajectories slide along the opposite curve only once before being chaotically reflected into the stadium. But this behavior is sufficient for sending a message from one end of the stadium to the other.

The intersections x_n of the *periodic* whispering gallery orbits with the x-axis are plotted in Figure 5 as well. It can be seen that the system of tulips approaches this series for increasing values of n.

4. Conclusion

The whispering gallery map introduced in this paper visualizes the connection between the isolated whispering gallery orbits and the manifold of all other possible trajectories inside the Bunimovich stadium. This map, unlike many other methods of dealing with chaotic systems, allows a survey of the dynamics for
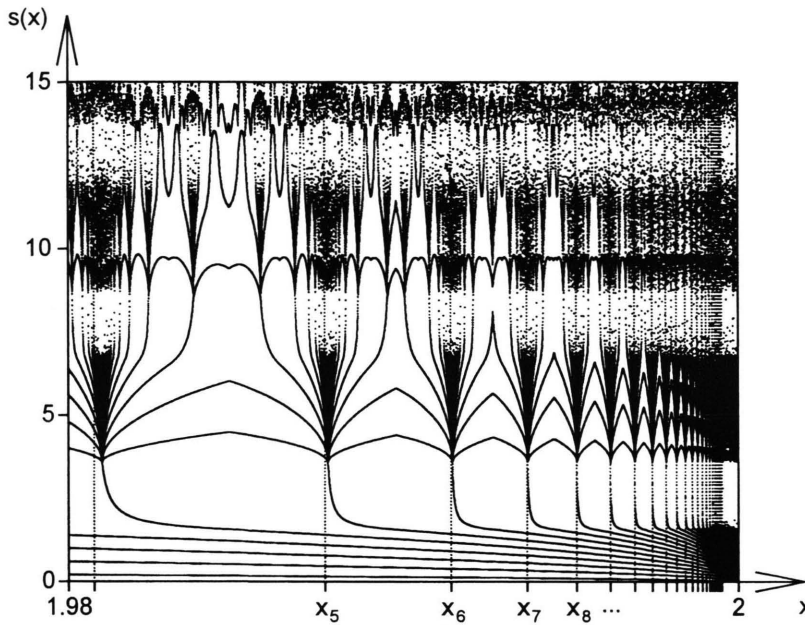
Fig. 5. Section of the system of virtual image walls seen by an observer sliding along the x-axis. The series of "tulip"-like channels corresponds to the whispering gallery orbits.

all possible starting conditions. The resulting virtual multifractals possess fascinating structures featuring self-similarities and locally varying roughness. They are rewarding objects for further mathematical investigations.

Considering whispering galleries, a 3-D generalization of the mirror cabinet idea appears to be promising. For example, inside the sphere all whispering gallery orbits emitted from a point at the wall will travel along meridians and will all meet each other at a "pole" on the opposite side of the sphere [7].

Therefore, the whispering gallery effect ought to be dramatically enhanced for the antipode.

Acknowledgement

[1] Lord Rayleigh, The Theory of Sound, No 287, Macmillan, New York 1894, reprinted by Dover, New York 1945.
[2] Lord Rayleigh, Proc. Roy. Inst., 15 January 1904, reprinted in Scientific Papers, Vol. V, p. 171, Dover, New York 1964.
[3] M. V. Berry, Proc. Roy. Soc. London **A 413**, 183 (1987).
[4] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer, New York 1990.
[5] C. A. Kruelle, A. Kittel, and R. P. Huebener, Z. Naturforsch. **48a**, 1039 (1993).
[6] C. A. Kruelle, A. Kittel, J. Peinke, R. Richter, and R. P. Huebener, Physica **D 102**, 227 (1997).
[7] J.-D. Polack, Sciences et Avenir, April 1995, p. 32.